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Solvable Models of Classical Lattice Gases

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We study classical lattice gases at fixed temperature but variable fugacity. It is shown how the thermodynamic functions may be calculated exactly provided the Boltzmann weights are representable as principal minors of a convolution operator. We explicitly construct this operator for the cluster models of Fisher and Felderhof.

KEY WORDS: Lattice gas; correlation functions; Ising model; Fisher–Felderhof model.

1. INTRODUCTION

The general theory of lattice gases as developed by Gallavotti, Dobrushin, Ruelle, Miracle Sole, Lanford and Robinson, Israel, and others has revealed a beautiful mathematical structure with a wide range of applications. This work evidently paved the way for a deeper understanding of equilibrium statistical mechanics and dynamical systems alike. The interested reader is referred to the reviews by Israel,(1) Ruelle,(2) Georgii,(3) and Mayer.(4)

Unfortunately, very few exactly solvable models are known. To find more, we need to understand why the existing models lead to explicit formulas for macroscopic quantities. (We would say for example that the pressure of the infinitely extended gas is explicitly known if it can be written as an integral over the first Brillouin zone.)

The present investigation arose from an attempt to generalize conditions satisfied by the one-dimensional cluster models of Fisher and Felderhof.(5) As is known,(6) these models admit explicit solutions not only in the continuum but also on a lattice. Comparison shows that the physics is almost the same, but there are simplifying features for the lattice version.

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A substantial portion of our discussion does not need the postulate that the lattice be one-dimensional. Thus, whenever possible we use $\mathbb{Z}^\nu$ as the underlying lattice. We thereby hope to stimulate the search for ν-dimensional solvable models. However, we confine ourselves to the special case where each lattice site may be occupied by either one or no particle. In the spin language, this is the familiar Ising model.

2. CONVOLUTION OPERATORS AND TOEPLITZ DETERMINANTS

With the infinitely extended lattice $\mathbb{Z}^\nu$ we associate the Hilbert space of square summable functions, $L^2(\mathbb{Z}^\nu)$, on which we wish to consider convolution operators,

$$(V\xi)(x) = \sum_y v(x-y)\xi(y) \qquad (x, y \in \mathbb{Z}^\nu) \tag{2.1}$$

Clearly, V may also be viewed as an infinite matrix $V_{x,y} = v(x-y)$ indexed by $\mathbb{Z}^\nu \times \mathbb{Z}^\nu$. Let us recall the definition of a principal minor

$$\det_X V = \det(V_{x,y}) \qquad (x, y \in X) \tag{2.2}$$

where X is any finite subset of $\mathbb{Z}^\nu$. We assign the value 1 to $\det_\varnothing V$. In a sense, these minors generalize the concept of a Toeplitz determinant. We wish to write

$$e^{-U(X)} = \det_X V \tag{2.3}$$

where we interpret $U(X)$ as the energy function of a lattice gas and X as the set of occupied sites. We therefore require $\det_X V > 0$ for all X. At first sight, (2.3) might seem an ad hoc definition. However, as will be discussed in Section 4, the Fisher–Felderhof models (including the model that Ising solved in 1925) share this peculiar structure which we believe has a fundamental significance. It will become less mystical as we progress.

We must not expect $U(X)$ to always describe pair potentials and short-range interactions. It will be seen later how special properties of the energy $U(X)$ are reflected in the operator V. Implicitly in (2.3) is the assumption $U(\varnothing) = 0$ and the choice of "natural" boundary conditions: there is nothing outside X to interact with the particles in X. The definition (2.3) also takes care of the translation invariance.

Let X denote the number of occupied sites and let z be the fugacity of the system. Then the grand partition function for a finite volume $\Lambda \subset \mathbb{Z}^\nu$ is

$$\Xi_\Lambda(z) = \sum_{X \subset \Lambda} z^{|X|} e^{-U(X)} = \det_\Lambda(1 + zV) \tag{2.4}$$

This formula is easily derived within the framework of exterior algebra (see the Appendix). In order to study the thermodynamic limit we need to assume $\sum|v(x)| < \infty$. Then

$$v(x) = \int dp\, e^{-ipx}\tilde{v}(p) \tag{2.5}$$

where the Fourier transform $\tilde{v}$ is a bounded function on the torus $-\pi \leqslant p_k < \pi$ $(k = 1, \ldots, \nu)$ with measure $dp = (2\pi)^{-\nu}dp_1 \cdots dp_\nu$. Here and henceforth integrations with respect to p are over the torus (i.e., the first Brillouin zone). We shall also impose the condition

$$\overline{\tilde{v}(p)} = \tilde{v}(-p) \tag{2.6}$$

which renders $v(x)$ a real function.

Notice that the Fourier decomposition (2.5) provides a spectral decomposition of the convolution operator V. By the functional calculus, any reasonable function f of V is a convolution operator arising from $f(\tilde{v}(p))$. Notice also that if z is complex and confined to the open disk $|z| < \inf|\tilde{v}(p)|^{-1}$ the operator $Q = \log(1 + zV)$ is well defined as has diagonal elements $Q_{x,x} = P(z)$, where

$$P(z) = \int dp \log[1 + z\tilde{v}(p)] \tag{2.7}$$

independent of the site x. As is seen from (2.4), $\log \Xi_\Lambda(z)$ becomes the trace of Q for a large volume Λ. This then identifies $P(z)$ as the pressure of the system. It may be worth mentioning that the reality condition (2.6) allows us to rewrite (2.7) if z is real:

$$P(z) = \int dp \log|1 + z\tilde{v}(p)| \tag{2.8}$$

From (2.7) we obtain the particle density

$$\rho(z) = z\frac{d}{dz}P(z) = \int dp \left\{1 + [z\tilde{v}(p)]^{-1}\right\}^{-1} \tag{2.9}$$

which extends to an analytic function of the fugacity z except on the singular set

$$M = \{z \in \mathbb{C} \mid z\tilde{v}(p) = -1 \text{ for some } p\} \tag{2.10}$$

A phase transition occurs if M intersects the positive real axis. A special situation of interest arises if $|\tilde{v}(p)| = 1$ for all p. In this case M is seen to be a subset of the Lee–Yang circle $|z| = 1$ and V is an infinite orthogonal matrix. The significance of this situation will become clear when we adopt the spin language and study spin correlations.

3. CORRELATION FUNCTIONS

We show how to obtain explicit expressions for the correlation functions. With any finite $A \subset \Lambda$ we associate an Ising variable

$$\sigma_A(x) = (-1)^{|X \cap A|} \tag{3.1}$$

The expectation in the grand canonical ensemble, $\langle \sigma_A \rangle_\Lambda$, is given by

$$\begin{aligned} \Xi_\Lambda(z)\langle \sigma_A \rangle_\Lambda &= \sum_{X \subset \Lambda} \sigma_A(X) z^{|X|} e^{-U(X)} \\ &= \sum_{X \subset \Lambda} \det_X(zJ_A V) \\ &= \det_\Lambda(1 + zJ_A V) \end{aligned} \tag{3.2}$$

where we introduced the following involution on $L^2(\mathbb{Z}^\nu)$:

$$(J_A \xi)(x) = \begin{cases} -\xi(x), & x \in A \\ \xi(x), & x \notin A \end{cases} \tag{3.3}$$

Thus we are left with the problem of unraveling the expression (3.2) with regard to its dependence on Λ and A. To this end we introduce the projection operator $P_\Lambda = \frac{1}{2}(1 - J_\Lambda)$ and write $R_\Lambda = P_\Lambda R P_\Lambda$ for any operator R. Suppose now that three operators R, S, T satisfy the equation

$$1 + R_\Lambda = (1 + S_\Lambda)(1 + T_\Lambda) \tag{3.4}$$

Then

$$\det_\Lambda(1 + R) = \det_\Lambda(1 + S)\det_\Lambda(1 + T) \tag{3.5}$$

Likewise, we write $J_A = 1 - 2P_A$ in (3.2). It is easily verified that the operators $R = zJ_A V$, $S = -2zP_A V_\Lambda(1 + zV_\Lambda)^{-1}$, and $T = zV$ satisfy (3.4) and hence (3.5). Insertion in (3.2) yields

$$\begin{aligned} \langle \sigma_A \rangle_\Lambda &= \det_\Lambda\left[1 - 2zP_A V_\Lambda(1 + zV_\Lambda)^{-1}\right] \\ &= \det_A\left[1 - 2zV_\Lambda(1 + zV_\Lambda)^{-1}\right] \\ &= \det_A \frac{1 - zV_\Lambda}{1 + zV_\Lambda} \end{aligned} \tag{3.6}$$

where we used a simple property of the determinant, i.e., for any operator R and any subset $A \subset \Lambda$

$$\det_\Lambda(1 + P_A R) = \det_\Lambda(1 + P_A R P_A) = \det_A(1 + R) \tag{3.7}$$

In the thermodynamic limit, we obtain $\langle \sigma_A \rangle$ as a generalized Toeplitz determinant,

$$\langle\sigma_A\rangle = \det_A W \tag{3.8}$$

where

$$W = \frac{1 - zV}{1 + zV} \tag{3.9}$$

which is a convolution operator, $W_{x,y} = w(x-y)$, arising from

$$\tilde{w}(p) = \frac{1 - z\tilde{v}(p)}{1 + z\tilde{v}(p)} \tag{3.10}$$

Since $\tilde{w}(p) = \tilde{w}(-p)$, $w(x)$ is a real function on $\mathbb{Z}^\nu$. In the above derivation we tacitly assumed that z stays away from the critical values. As a result we obtained a unique equilibrium state of the infinite system. From this state, as z approaches a critical value (on the positive real axis), we may obtain two different states depending on whether our approach is from above or below. This is due to the fact that $\tilde{w}(p)$ develops a singularity as a function of p.

Let us now investigate what is implied by our basic formula (3.8). First, take $A = \{x\}$:

$$\langle\sigma_x\rangle = w(0) = \int dp\, \tilde{w}(p) \tag{3.11}$$

which agrees, of course, with (2.9) since magnetization is related to the density by $\langle\sigma_x\rangle = 1 - 2\rho$. Next, take $A = \{x, y\}, x \neq y$:

$$\langle\sigma_x\sigma_y\rangle = w(0)^2 - w(x-y)w(y-x) \tag{3.12}$$

This shows that $w(x)$ has to do with the truncated two-point function.

What is the simplifying feature of the condition $|\tilde{v}(p)| = 1$? In this case we would write $z = e^{-2B}$ and call B the "magnetic field." Let us distinguish the following two cases:

(1) The range of $\tilde{v}(p)$ includes -1. Then $B = 0$ is a critical value. This suggests a study of the magnetization $m(B) = \langle\sigma_x\rangle$ as a function of the magnetic field:

$$m(B) = \int dp \frac{e^{2B} - \tilde{v}(p)}{e^{2B} + \tilde{v}(p)} \tag{3.13}$$

Notice that $m(-B) = -m(B)$. Changing variables to

$$f(p) = \left|\frac{1 + \tilde{v}(p)}{1 - \tilde{v}(p)}\right|, \qquad \epsilon = \tanh B \tag{3.14}$$

we may write

$$m(B) = \epsilon + (1 - \epsilon^2) \int dp \frac{\epsilon}{f(p)^2 + \epsilon^2} \tag{3.15}$$

to obtain the spontaneous magnetization

$$m(+0) = \pi \int dp\, \delta(f(p)) \tag{3.16}$$

provided $f(p)$ is a sufficiently smooth function. An elementary formula avoiding δ functions is the following:

$$m(+0) = \lim_{E \downarrow 0} \pi E^{-1} \int_{f(p) \leqslant E} dp \tag{3.17}$$

2. The range of $\tilde{v}(p)$ does not include -1. Then $B = 0$ is not a critical value. At $B = 0$, W is just the Cayley transform of the orthogonal matrix V, hence is antisymmetric proving $w(-x) = -w(x)$. As a result, spin correlation functions of odd order vanish identically, while correlation functions of even order are nonnegative (as squares of Pfaffians). The static susceptibility of this system is

$$\chi = \sum_x \langle \sigma_x \sigma_0 \rangle = 1 + \int dp \left| \frac{1 - \tilde{v}(p)}{1 + \tilde{v}(p)} \right|^2 \tag{3.18}$$

4. THE FISHER–FELDERHOF MODELS

We now specialize to one-dimensional translation invariant lattice systems and characterize the Fisher–Felderhof (FF) models by the following property:

Separation Property. If some site $n \in \mathbb{Z}$ is not occupied by a particle, the configuration on the left does not interact with the configuration on the right, that is to say

$$U(X) = U(X_l) + U(X_r) \tag{4.1}$$

if $n \notin X$ and $X_l = \{x \in X \mid x < n\}$, $X_r = \{x \in X \mid x > n\}$.

In some sense, this extends the concept of nearest-neighbor interaction to many-body forces. Recall that one may recursively define a potential $\phi(X)$ such that $\phi(\emptyset) = 0$ and

$$U(X) = \sum_{Y \subset X} \phi(Y) \tag{4.2}$$

Using this device we may also characterize the FF models by following the equivalent property:

Cluster Property. $\phi(X) = 0$ unless X is a cluster (i.e., an interval of integers).

If X is a cluster of size $|X| = n > 0$, we write $E_n = U(X)$ and $J_n = \phi(X)$. From (4.2) we have that

$$E_n = \sum_{k=1}^{n} (n - k + 1)J_k \tag{4.3}$$

The model under consideration is completely specified by giving either $\{E_n\}$ or $\{J_n\}$, i.e., a denumerable set of real numbers. We then define the "master function"

$$F(u) = 1 + \sum_{n=1}^{\infty} u^n e^{-E_n} \tag{4.4}$$

and require that the series has a nonzero radius of convergence: $\lim n^{-1}E_n > -\infty$. There is still another way to characterize the FF models.

Theorem. $U(X)$ has the separation property if and only if

$$e^{-U(X)} = \det_X V \tag{4.5}$$

where $V_{n,m} = v(n - m)$ and $v(n) = 0$ for $n < -1$.

Proof. The "if" part is fairly obvious. Any finite set $X \subset \mathbb{Z}$ may be uniquely decomposed into clusters C_i and

$$\det_X V = \prod_i \det_{C_i} V \tag{4.6}$$

owing to the fact that $V_{n,m} = 0$ for $n + 1 < m$.

To prove the "only if" part we consider, for instance, a cluster of size n and obtain

$$\begin{vmatrix} v(0) & v(-1) & & 0 \\ v(1) & v(0) & \ddots & \\ \vdots & \ddots & \ddots & v(-1) \\ v(n-1) & \cdots & v(1) & v(0) \end{vmatrix} = e^{-E_n} \tag{4.7}$$

This equation determines $v(n-1)$ uniquely provided the $v(k)$, $k = 0, \ldots, n-2$, have been determined previously. If, in (4.7), we let $n = 1, 2, \ldots,$ all $v(n)$ may be calculated in a recursive manner provided $v(-1)$ has been assigned a certain value to begin with. Thus we fix $r = -v(-1)^{-1} > 0$ and look at the formal power series

$$B(\zeta) = \sum_{n=0}^{\infty} v(n-1)\zeta^n \tag{4.8}$$

As Wronski observed,[7] (4.7) provides a scheme to find explicitly the coefficients of the inverse power series $1/B(\zeta)$ which in our case is $-rF(r\zeta)$. Conversely, $v(n)$ may be written as a Toeplitz determinant,

$$\begin{vmatrix} re^{-E_1} & & 1 & 0 \\ r^2e^{-E_2} & & re^{-E_1} & \ddots \\ \vdots & \ddots & \ddots & 1 \\ r^{n+1}e^{-E_{n+1}} & \cdots & r^2e^{-E_2} & re^{-E_1} \end{vmatrix} = (-1)^n rv(n) \tag{4.9}$$

which proves our theorem. ■

Let us now establish the connection between $B(\zeta)$ and $\tilde{v}(\alpha)$ where $\zeta = e^{i\alpha}$. The function $F(r\zeta)$ has an absolutely convergent Fourier series provided r is smaller than the radius of the defining series. Moreover, if the function $F(r\zeta)$ does not vanish on the unit circle $|\zeta| = 1$, then by virtue of a well-known theorem of Wiener on division by a Fourier series, the inverse

$$B(\zeta) = \left[-rF(r\zeta)\right]^{-1} \tag{4.10}$$

has an absolutely convergent Fourier series with Fourier coefficients given by $v(n-1)$. It remains to show that r may be chosen such that $F(r\zeta)$ does not vanish on $|\zeta| = 1$. In fact, $F(r)$ is an increasing convex function in $r > 0$ and thus $F(r) = 2$ has a unique solution $r = R$. If $r < R$,

$$\begin{aligned} |F(r\zeta)| &\geq 1 - \left|\sum_n e^{-E_n}(r\zeta)^n\right| \\ &> 1 - \sum_n e^{-E_n}R^n = 2 - F(R) \end{aligned} \tag{4.11}$$

hence $|F(r\zeta)| > 0$. This establishes

$$\tilde{v}(\alpha) = \sum_{n=-1}^{\infty} v(n)e^{in\alpha} = \left[-re^{i\alpha}F(re^{i\alpha})\right]^{-1} \tag{4.12}$$

Appealing to our general formula (2.7) we obtain the pressure as a function of the fugacity,

$$P(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \log\left[1 - \frac{z}{re^{i\alpha}F(re^{i\alpha})}\right] \tag{4.13}$$

which may also be written as a Cauchy integral,

$$P(z) = \frac{1}{2\pi i}\int_{|u|=r} \frac{du}{u}\log\left[1 - \frac{z}{uF(u)}\right] \tag{4.14}$$

We restrict the fugacity $z > 0$ such that $u_z < r$ where u_z is the unique positive solution of

$$u_z F(u_z) = z \tag{4.15}$$

Notice that u_z is a monotone function of the fugacity and $u_z < z$.

To evaluate the integral (4.14) we proceed in two steps: (1) We show that, as a function of u, $uF(u) - z$ has no zeros in the disk $|u| \leqslant u_z$ besides the simple zero at $u = u_z$. (2) Likewise, $F(u)$ does not vanish in $|u| \leqslant u_z$ since $u_z < R$ [see (4.11)]. We may write

$$1 - \frac{z}{uF(u)} = z\left(\frac{1}{u_z} - \frac{1}{u}\right)f(u) \tag{4.16}$$

There exists $\epsilon > 0$ such that $f(u)$ is analytic and nonzero in the open disk $|u| < u_z + \epsilon$. Of course, f depends on z, but notice that $f(0) = 1$ since $F(p) = 1$. Therefore, $u^{-1}\log f(u)$ is analytic and does not contribute to the Cauchy integral (4.13) if $u_z < r < u_z + \epsilon$ leaving

$$P(z) = \frac{1}{2\pi i}\int_{|u|=r}\frac{du}{u}\log\left(\frac{z}{u_z} - \frac{z}{u}\right) = \log\frac{z}{u_z} \tag{4.17}$$

It remains to prove the following lemma.

Lemma. $uF(u) - z$ has no zero in the domain $|u| \leqslant u_z$, $u \neq u_z$.

Proof. For any $a > 0$,

$$\begin{aligned} |z - uF(u)| &\geqslant |z + a - u| - |a - u + uF(u)| \\ &\geqslant |z + a - u| - a - |u|\,|F(u) - 1| \end{aligned}$$

Now, $|u| \leqslant a$ implies $|F(u) - 1| \leqslant F(a) - 1$ since $F(u) - 1$ has a power series with positive coefficients. Moreover, $aF(a) = z$ if $a = u_z$ and thus

$$|z - uF(u)| \geqslant |z + u_z - u| - z \tag{4.18}$$

To complete the argument we notice that the disk $|z + u_z - u| \leqslant z$ intersects the disk $|u| \leqslant u_z$ in precisely one point, $u = u_z$. Therefore, the right-hand side of (4.18) is strictly positive granted the assumptions on u. ■

As has been shown by Fisher and Felderhof, the FF models may exhibit liquid–gas phase transitions due to the presence of n-body interactions of arbitrarily large n. However, these transitions are of a very special nature and are marked by the fact that the liquid has the close packing density $\rho = 1$ (unless $E_n = +\infty$ for some n). This behavior may be studied in examples like the logarithmic model where $E_n = J\log n$ and hence

$$F(u) = 1 + \sum_{n=1}^{\infty} u^n n^{-J} \tag{4.19}$$

Here the transition occurs at $J = 1$, $\rho = 1$. Above the critical coupling, the gas condenses at a finite pressure

$$P = \log\left[1 + \zeta(J)\right] \tag{4.20}$$

(ζ is Riemann's zeta function) whereas below the critical coupling the pressure rises to infinity as ρ tends to 1. It may be worthwhile to point out that the one-dimensional Ising model,

$$U = J\sum_n (1 - \sigma_n\sigma_{n+1}) + B\sum_n (1 - \sigma_n) \tag{4.21}$$

appears as the following special case of the FF models:

$$E_n = 4J, \qquad z = e^{-2B} \tag{4.22}$$

In words, a cluster of size n creates two dissatisfied bonds with energy $2J$ each. Thus, the master function takes on the simple form

$$F(u) = \frac{1 - a^2 u}{1 - u} \qquad (|u| < 1) \tag{4.23}$$

where

$$a^2 = 1 - e^{-4J} \tag{4.24}$$

To reproduce the standard textbook formula for the free energy we would simply determine $\log(z/u_z)$ where u_z is the positive solution of the quadratic equation $uF(u) = z$. Instead, we consider the more interesting problem of investigating the convolution operator V for ferromagnetic coupling, $J > 0$.

From (4.11) we infer that

$$\tilde{v}(\alpha) = \sum_n v(n)e^{in\alpha} = \frac{1 - ae^{-i\alpha}}{1 - ae^{i\alpha}} \tag{4.25}$$

provided we choose $r = 1/a$. Then $|\tilde{v}(\alpha)| = 1$ and $\tilde{v}(-\alpha) = 1/\tilde{v}(\alpha)$. Setting $V_{n,m} = v(n - m)$ as before, we obtain an infinite orthogonal matrix with

$$v(n) = \begin{cases} a^n(1 - a^2), & n \geqslant 0 \\ -a, & n = -1 \\ 0, & n \leqslant -2 \end{cases} \tag{4.26}$$

One may also look at the function

$$\tilde{w}(\alpha) = \frac{1 - \tilde{v}(\alpha)}{1 + \tilde{v}(\alpha)} = i\frac{a\sin\alpha}{1 - a\cos\alpha} = \sum_n w(n)e^{in\alpha} \tag{4.27}$$

that generates the correlation functions at $B = 0$,

$$\langle\sigma_A\rangle = \det w(n - m) \qquad (n, m \in A) \qquad (A \subset \mathbb{Z}). \tag{4.28}$$

It follows from (4.26) that

$$w(n) = \begin{cases} b^n, & n > 0 \\ 0, & n = 0 \\ -b^{-n}, & n < 0 \end{cases} \tag{4.29}$$

where $2b = a(1 + b^2)$, hence $b = (\tanh J)^{1/2}$.

5. UNSOLVED PROBLEMS

We described a class of lattice models using the criterion that there exists a convolution operator V from which the energy function U is derived: $\det_X V = \exp[-U(X)]$. This was done for a fixed temperature, say $\beta = 1$. We did not discuss how a change of temperature affects the operator V. Not even the existence of $V(\beta)$ such that $\det_X V(\beta) = \exp[-\beta U(X)]$ has been shown. We also failed to obtain the most general solution of the positivity requirement $\det_X V > 0$. Of course, these problems do not arise in the FF models because there we found a way to invert the relation between V and U. To sum up we formulate our main problem:

Main Problem. Under what conditions can V be obtained from U?

In principle, one should be able to decide whether the two-dimensional Onsager model falls within the described category. Suppose there were a method to construct the convolution operator for this model at $B = 0$; this then would at the same time solve the difficult problem with $B \neq 0$. Notice, however, that the two-dimensional Onsager model with $B = 0$ is intimately related to a combinatorial problem involving dimers which has a solution in terms of Pfaffians rather than of determinants. Does this mean that our approach to the problem is bound to fail?

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APPENDIX

Let $L^2(\mathbb{Z}^\nu)$ be the Hilbert space of square summable complex functions on $\mathbb{Z}^\nu$ and let $e_x(y) = \delta_{x,y}$ for $x, y \in \mathbb{Z}^\nu$. By construction, the functions e_x form a basis. Given a bounded linear operator V on this space, it is uniquely determined by its matrix elements $V_{x,y} = (e_x, Ve_y)$. With any finite subset $A \subset \mathbb{Z}^\nu$ we associate a subspace E_A spanned by the vectors e_x with $x \in A$. Let V_A be the restriction of V to this subspace. Then V_A has matrix elements $V_{x,y}$ where $x, y \in A$. We now extend V_A to a linear operator $\Lambda(V_A)$ on the exterior algebra $\Lambda(E_A)$. To obtain the matrix representation for $\Lambda(V_A)$ we first construct a basis for $\Lambda(E_A)$. Assume for a moment that A has been totally ordered (for instance by lexicographic ordering of $\mathbb{Z}^\nu$). For any $X \subset A$ we would write

$$e_X = e_x \wedge e_y \wedge \cdots \wedge e_z \tag{A.1}$$

where $(x, y, \ldots, z)$ is the sequence of elements in X arranged in increasing order. We put $e_\varnothing = 1$. Any vector $u \in \Lambda(E_A)$ has an expansion $\sum u_X e_X$. The scalar product $(u, v) = \sum \bar{u}_X v_X$ gives $\Lambda(E_A)$ the structure of a Hilbert space, also known as the Fock space (with Fermi statistics) over E_A. The remarkable fact about this construction is that

$$(e_X, \Lambda(V_A) e_X) = \det_X V \qquad (X \subset A) \tag{A.2}$$

i.e., the diagonal elements of $\Lambda(V_A)$ considered as a matrix coincide with the principal minors of V. The second fact about this construction is the validity of the trace formula[8]

$$\begin{aligned}\sum_{X \subset A} z^{|X|} \det_X V &= \sum_{X \subset A} \det_X(zV) = \operatorname{tr} \Lambda(zV_A) \\ &= \det(1 + zV_A) = \det_A(1 + zV)\end{aligned} \tag{A.3}$$

It is easily checked that the ordering of $\mathbb{Z}^\nu$ is, in fact, irrelevant for the diagonal elements of $\Lambda(V_A)$.

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